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EXPLORING THE SUM-CONNECTIVITY ENERGY OF GRAPHS: NEW RESULTS

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Abstract

Let G be a graph of order n with the vertex set $\{v_1, v_2, v_3, \cdots, v_n\}$, and d_i be the degree of vertex v_i for all $i \in \{1, 2, 3, \cdots, n\}$, then the sum-connectivity matrix of a graph G is the $n \times n$ square matrix denoted and defined by $M_{SC}(G) = [s_{ij}]_{n\times n}$, where, $s_{ij} = \frac{1}{\sqrt{d_i+d_j}}$, if vertices v_i and v_j are adjacent and $s_{ij} = 0$, otherwise. The sum-connectivity energy is the sum of absolute values of the eigenvalues of $M_{SC}(G)$. In this paper, the sum-connectivity energy of some graphs are investigated.

Keywords: Eigenvalue, Graph Energy, Sum-Connectivity Matrix, Sum-Connectivity Energy

AMS Subject Classification(2020): 05C50,05C76

1 Introduction

Let G be a simple, undirected and connected graph with vertex set $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$. The degree of a vertex v_i is denoted by d_i, is the number of vertices adjacent to a vertex v_i, for all $i \in \{1, 2, 3, \cdots, n\}$.

Definition 1.1. *The m-Shadow graph, $D_m(G)$ of a connected graph G is constructed by taking m copies of G say $G_1, G_2, ..., G_m$, then join each vertex u in G_i to the neighbours of the corresponding vertex v in G_j,* $1 \leq i, j \leq m$.

Definition 1.2. *The m-Splitting graph, $Spl_m(G)$ of a connected graph G is constructed by adding to each vertex v a new m vertices, say $v_1, v_2, ..., v_m$ such that v_i, $1 \leq i \leq m$ is adjacent to every vertex that is adjacent to v is G.*

Definition 1.3. *The double cover of the graph G with vertex set $\{v_1, v_2, ..., v_n\}$ is the bipartite graph G' with bipartition (X, Y), $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$, where x_i and y_j are adjacent in G' if and only if v_i and v_j are adjacent in the graph G.*

The adjacency matrix of a graph G is a symmetric matrix of order n and defined as $A(G) = [a_{ij}]_{n \times n}$, where,

$$a_{ij} = \begin{cases} 1 & ;\text{if } v_i \text{ and } v_j \text{are adjacent} \\ 0 & ;\text{if } v_i \text{ and } v_j \text{are non} - \text{adjacent} \end{cases}$$

Let $\alpha_1, \alpha_2, ..., \alpha_n$ be the eigenvalues of the adjacency matrix, $A(G)$, then the energy of a graph G, $\mathscr{E}(G)$ is the sum of absolute value of eigenvalues of adjacency matrix, $A(G)$ of a graph G with their multiplicity. *i.e.*

$$\mathscr{E}(G) = \sum_{i=1}^{n} |\alpha_i|$$

The concept of energy was introduced by Gutman [4] in 1978.

The concept of graph energy has the connection between the graph theory and the approximation of the total π-electron energy of a conjugated hydrocarbon in molecular chemistry [8]. A conjugated hydrocarbon can be represented by a graph called a molecular graph in which each carbon atom is represented by a vertex, carbon-carbon bond by an edge and hydrogen atoms are not included. The study of molecular structure using the energy of its graph is known as chemical graph theory. In short, the graph energy is useful to find the energy of organic compounds in Chemistry.

There are many variants of graph energy like Distance energy, Randić energy, Harary energy and Laplacian energy are also available in the literature [1, 2, 3, 6, 9, 11, 12, 13, 14].

The concept of sum-connectivity energy was introduced by Zhou and Trinajstic [16] in 2010.

The sum-connectivity matrix of a graph G is a symmetric matrix of order $n \times n$ and defined as $M_{SC}(G) = [s_{ij}]_{n\times n}$, where,

$$s_{ij} = \begin{cases} \dfrac{1}{\sqrt{d_i + d_j}} & ; \text{if } v_i \text{ and } v_j \text{are adjacent} \\ 0 & ; \text{if } v_i \text{ and } v_j \text{are non} - \text{adjacent} \end{cases}$$

The eigenvalues $\vartheta_1, \vartheta_2, ..., \vartheta_n$ of the sum-connectivity matrix, $M_{SC}(G)$ is known as sum-connectivity eigenvalues of the graph G and the sum-connectivity spectrum of the graph G is denoted by $spec_{SC}(G)$. The *sum-connectivity energy* of the graph G, $\mathscr{E}_{SC}(G)$ is the sum of absolute values of sum-connectivity eigenvalues of a graph G with their multiplicity. *i.e.*

$$\mathscr{E}_{SC}(G) = \sum_{i=1}^{n} |\vartheta_i|$$

Illustration 1.4. *In Fig. 1, the cycle C_4 is shown in which the set of vertices is $\{v_1, v_2, v_3, v_4\}$.*

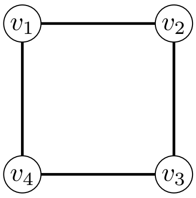

Figure 1: Cycle C_4 with the vertices $\{v_1, v_2, v_3, v_4\}$

$$\text{Then, } M_{SC}(C_4) = \begin{array}{c c} & \begin{array}{cccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \boldsymbol{v_4} \end{array} \\ \begin{array}{c} \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \boldsymbol{v_4} \end{array} & \begin{pmatrix} 0 & \frac{1}{\sqrt{4}} & 0 & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} & 0 & \frac{1}{\sqrt{4}} & 0 \\ 0 & \frac{1}{\sqrt{4}} & 0 & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} & 0 & \frac{1}{\sqrt{4}} & 0 \end{pmatrix}_{4\times 4} \end{array}$$

$$\text{Thus, } spec_{SC}(C_4) = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

Hence, $\mathscr{E}_{SC}(C_4) = 2(0) + 1 \cdot |1| + |-1| = 2$

In this paper, the sum-connectivity energy of regular graph, m-shadow, m-splitting and double cover of graphs are investigated.

For standard terminology and notations in graph theory, rely upon West [15] while for any undefined term related to energy of graphs, refer to Gutman [5, 10].

For our ready reference, some existing results are stated below.

Proposition 1.5. *[7] Let $A = [a_{ij}]$ be the matrix of size $m \times n$ and B be the matrix of size $p \times q$, then the Kronecker product of the matrix A and B is denoted by $A \otimes B$ and defined as $A \otimes B = \Big(a_{ij}B\Big)_{mp\times nq}$.*

Proposition 1.6. *[7] If λ is an eigenvalue of the matrix $A = [a_{ij}]_{n\times n}$ with corresponding eigenvector x and μ is an eigenvalue of the matrix $B = [b_{ij}]_{m\times m}$ with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

Proposition 1.7. *[7] Let*

$$M = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

be a symmetric matrix. Then the spectrum of M is the union of spectrum of $A + B$ and $A - B$.
i.e. $spec(M) = spec(A + B) \cup spec(A - B)$.

2 Main Results

Theorem 2.1. *Let G be a r-regular graph with $\alpha_1, \alpha_2, ..., \alpha_n$ eigenvalues, then*

$$\mathscr{E}_{SC}(G) = \frac{1}{\sqrt{2r}}\mathscr{E}(G)$$

Proof. Let G be a r-regular graph of order n with the vertex set $\{v_1, v_2, ..., v_n\}$, then the sum-connectivity matrix $M_{SC}(G)$ is defined as

$$M_{SC}(G) = \frac{1}{\sqrt{2r}}A(G)$$

where, $A(G)$ is the adjacency matrix of the graph G. Hence, the sum-connectivity energy of the r-regular graph G,

$$\mathscr{E}_{SC}(G) = \frac{1}{\sqrt{2r}}\mathscr{E}(G)$$

$\square$

Theorem 2.2. *For the graph G, $\mathscr{E}_{SC}(D_m(G)) = \sqrt{m}\mathscr{E}_{SC}(G)$.*

Proof. Let G be a graph of order n with the vertex set $\{v_1, v_2, ..., v_n\}$, then the sum-connectivity matrix $M_{SC}(G)$ is given by

$$M_{SC}(G) = \begin{array}{c c} & \begin{array}{cccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} \end{array} \\ \begin{array}{c} \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \end{array} & \begin{pmatrix} 0 & \frac{1}{\sqrt{d_1+d_2}} & \frac{1}{\sqrt{d_1+d_3}} & \cdots & \frac{1}{\sqrt{d_1+d_n}} \\ \frac{1}{\sqrt{d_2+d_1}} & 0 & \frac{1}{\sqrt{d_2+d_3}} & \cdots & \frac{1}{\sqrt{d_2+d_n}} \\ \frac{1}{\sqrt{d_3+d_1}} & \frac{1}{\sqrt{d_3+d_2}} & 0 & \cdots & \frac{1}{\sqrt{d_3+d_n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{d_n+d_1}} & \frac{1}{\sqrt{d_n+d_2}} & \frac{1}{\sqrt{d_n+d_3}} & \cdots & 0 \end{pmatrix}_{n\times n} \end{array}$$

and the eigenvalues of the matrix $M_{SC}(G)$ are $\vartheta_1, \vartheta_2, ..., \vartheta_n$.

Now, to obtain m-shadow of the graph G, $D_m(G)$, consider m- copies $G_1, G_2, ..., G_m$ of the graph G and join each vertex of u of the graph G_i to the neighbors of the corresponding vertex v in the graph G_j, $1 \leq i, j \leq m$. Note that $N(v_i) = N(v_j^m)$. Then the sum-connectivity matrix of $D_m(G)$ is denoted by $M_{SC}(D_m(G))$ and defined as

$$\begin{aligned} M_{SC}(D_m(G)) &= \begin{pmatrix} \frac{1}{\sqrt{m}}M_{SC}(G) & \frac{1}{\sqrt{m}}M_{SC}(G) & \cdots & \frac{1}{\sqrt{m}}M_{SC}(G) \\ \frac{1}{\sqrt{m}}M_{SC}(G) & \frac{1}{\sqrt{m}}M_{SC}(G) & \cdots & \frac{1}{\sqrt{m}}M_{SC}(G) \\ \cdots & \cdots & \ddots & \cdots \\ \frac{1}{\sqrt{m}}M_{SC}(G) & \frac{1}{\sqrt{m}}M_{SC}(G) & \cdots & \frac{1}{\sqrt{m}}M_{SC}(G) \end{pmatrix}_{mn\times mn} \\ &= \begin{pmatrix} \frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \cdots & \frac{1}{\sqrt{m}} \\ \frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \cdots & \frac{1}{\sqrt{m}} \\ \cdots & \cdots & \ddots & \cdots \\ \frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \cdots & \frac{1}{\sqrt{m}} \end{pmatrix} \otimes M_{SC}(G) \\ &= \frac{1}{\sqrt{m}} J_m \otimes M_{SC}(G); \end{aligned}$$

where, J_m is the matrix of order m whose all entries are 1.

Further, we know that $spec(J_m) = \begin{pmatrix} m & 0 \\ 1 & m-1 \end{pmatrix}$

$\Rightarrow spec\left(\frac{1}{\sqrt{m}}J_m\right) = \begin{pmatrix} \sqrt{m} & 0 \\ 1 & m-1 \end{pmatrix}$

Therefore by Proposition 1.5,

$spec_{SC}(D_m(G)) = \begin{pmatrix} \sqrt{m}(\vartheta_i) & 0 \\ n & n(m-1) \end{pmatrix}$ Hence,

$$\begin{aligned} \mathscr{E}_{SC}(D_m(G)) &= \sum_{i=1}^{n} |\sqrt{m}(\vartheta_i)| \\ &= \sqrt{m}\sum_{i=1}^{n} |\vartheta_i| \\ \Rightarrow \mathscr{E}_{SC}(D_m(G)) &= \sqrt{m}\mathscr{E}_{SC}(G) \end{aligned}$$

□

Corollary 2.3. *Let G be a r-regular graph with $\alpha_1, \alpha_2, ..., \alpha_n$ eigenvalues, then*

$$\mathscr{E}_{SC}(D_m(G)) = \frac{1}{\sqrt{2mr}}\mathscr{E}(G)$$

Proof. Let G be a $r-$regular graph of order n with the vertex set $\{v_1, v_2, ..., v_n\}$.

Now, to obtain m-shadow of the r-regular graph G, $D_m(G)$, consider m-copies $G_1, G_2, ..., G_m$ of the graph G and join each vertex of u of the graph G_i to the neighbors of the corresponding vertex v in the graph G_j, $1 \leq i, j \leq m$. Moreover, $D_m(G)$ is mr-regular graph. Then by Theorem 2.1, the sum-connectivity matrix of $D_m(G)$ is denoted by $M_{SC}(D_m(G))$ and defined as

$$M_{SC}(D_m(G)) = \frac{1}{\sqrt{2mr}}A(G)$$

where, $A(G)$ is the adjacency matrix of a graph G. Hence, the sum-connectivity energy of the m-shadow of r-regular graph G,

$$\mathscr{E}_{SC}(D_m(G)) = \frac{1}{\sqrt{2mr}}\mathscr{E}(G)$$

□

Theorem 2.4. *For any graph G,*

$$\mathscr{E}_{SC}(Spl_m(G)) = \left(\sqrt{\frac{8m^2+m+1}{m(m+1)}}\right)\mathscr{E}_{SC}(G)$$

Proof. Let G be a graph of order n with the vertex set $\{v_1, v_2, ..., v_n\}$, then the sum-connectivity matrix $M_{SC}(G)$ is given by

$$M_{SC}(G) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \end{array} \begin{array}{c} \begin{array}{ccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} \end{array} \\ \begin{pmatrix} 0 & \frac{1}{\sqrt{d_1+d_2}} & \frac{1}{\sqrt{d_1+d_3}} & \cdots & \frac{1}{\sqrt{d_1+d_n}} \\ \frac{1}{\sqrt{d_2+d_1}} & 0 & \frac{1}{\sqrt{d_2+d_3}} & \cdots & \frac{1}{\sqrt{d_2+d_n}} \\ \frac{1}{\sqrt{d_3+d_1}} & \frac{1}{\sqrt{d_3+d_2}} & 0 & \cdots & \frac{1}{\sqrt{d_3+d_n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{d_n+d_1}} & \frac{1}{\sqrt{d_n+d_2}} & \frac{1}{\sqrt{d_n+d_3}} & \cdots & 0 \end{pmatrix}_{n\times n} \end{array}$$

and the eigenvalues of the matrix $M_{SC}(G)$ are $\vartheta_1, \vartheta_2, ..., \vartheta_n$.

Now, consider m-copies of vertex v_i, say $v_i^1, v_i^2, ..., v_i^m$, for $1 \leq i \leq n$ and join each vertex v_i^k for $1 \leq k \leq m$ to neighbors of the vertex v_i to obtain m-splitting of the graph G and is denoted by $Spl_m(G)$. Then the sum-connectivity matrix of $Spl_m(G)$ is denoted by $M_{SC}(Spl_m(G))$ and defined as

$$\begin{aligned}
M_{SC}(Spl_m(G)) &= \begin{pmatrix} \frac{1}{\sqrt{m}}M_{SC}(G) & \frac{2}{\sqrt{2m+2}}M_{SC}(G) & \cdots & \frac{2}{\sqrt{2m+2}}M_{SC}(G) \\ \frac{2}{\sqrt{2m+2}}M_{SC}(G) & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ \frac{2}{\sqrt{2m+2}}M_{SC}(G) & 0 & \cdots & 0 \end{pmatrix}_{mn\times mn} \\
&= \begin{pmatrix} \frac{1}{\sqrt{m}} & \frac{2}{\sqrt{2m+2}} & \cdots & \frac{2}{\sqrt{2m+2}} \\ \frac{2}{\sqrt{2m+2}} & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ \frac{2}{\sqrt{2m+2}} & 0 & \cdots & 0 \end{pmatrix}_{mn\times mn} \otimes M_{SC}(G) \\
&= M \otimes M_{SC}(G)
\end{aligned}$$

where, $M = \begin{pmatrix} \frac{1}{\sqrt{m}} & \frac{2}{\sqrt{2m+2}} & \cdots & \frac{2}{\sqrt{2m+2}} \\ \frac{2}{\sqrt{2m+2}} & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ \frac{2}{\sqrt{2m+2}} & 0 & \cdots & 0 \end{pmatrix}_{mn\times mn}$

Since, matrix M is of rank two. So, M has only two non-zero eigenvalues, say β and γ. Further,

$$\beta + \gamma = tr(M) = \frac{1}{\sqrt{m}} \tag{1}$$

Moreover,

$$\begin{aligned}
M^2 &= M \cdot M \\
&= \begin{pmatrix} \frac{1}{\sqrt{m}} & \frac{2}{\sqrt{2m+2}} & \cdots & \frac{2}{\sqrt{2m+2}} \\ \frac{2}{\sqrt{2m+2}} & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ \frac{2}{\sqrt{2m+2}} & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{m}} & \frac{2}{\sqrt{2m+2}} & \cdots & \frac{2}{\sqrt{2m+2}} \\ \frac{2}{\sqrt{2m+2}} & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ \frac{2}{\sqrt{2m+2}} & 0 & \cdots & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{m} + \frac{4m}{2m+2} & \frac{2}{\sqrt{m(2m+2)}} & \frac{2}{\sqrt{m(2m+2)}} & \cdots & \frac{2}{\sqrt{m(2m+2)}} \\ \frac{2}{\sqrt{m(2m+2)}} & \frac{4}{2m+2} & \frac{4}{2m+2} & \cdots & \frac{4}{2m+2} \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ \frac{2}{\sqrt{m(2m+2)}} & \frac{4}{2m+2} & \frac{4}{2m+2} & \cdots & \frac{4}{2m+2} \end{pmatrix}_{(m+1)\times(m+1)}
\end{aligned}$$

Therefore,

$$\beta^2 + \gamma^2 = tr(M^2) = \frac{1}{m} + \frac{8m}{2m+2} \tag{2}$$

Now, by solving equations (1) and (2), we get

$$\beta = \frac{\sqrt{m+1} + \sqrt{8m^2+m+1}}{2\sqrt{m(m+1)}} \text{ and } \gamma = \frac{\sqrt{m+1} - \sqrt{8m^2+m+1}}{2\sqrt{m(m+1)}}$$

Therefore, $spec_{SC}(M) =$

$$\begin{pmatrix} \frac{\sqrt{m+1} + \sqrt{8m^2+m+1}}{2\sqrt{m(m+1)}} & \frac{\sqrt{m+1} - \sqrt{8m^2+m+1}}{2\sqrt{m(m+1)}} & 0 \\ 1 & 1 & m-1 \end{pmatrix}$$

Since, $M_{SC}(Spl_m(G)) = M \otimes M_{SC}(G)$ and eigenvalues of $M_{SC}(G)$ are $\vartheta_1, \vartheta_2, ..., \vartheta_n$, then by Proposition 1.6,

$spec_{SC}(Spl_m(G)) =$

$$\begin{pmatrix} \frac{\sqrt{m+1} + \sqrt{8m^2+m+1}}{2\sqrt{m(m+1)}}\vartheta_1 & \frac{\sqrt{m+1} + \sqrt{8m^2+m+1}}{2\sqrt{m(m+1)}}\vartheta_2 & \cdots & \frac{\sqrt{m+1} + \sqrt{8m^2+m+1}}{2\sqrt{m(m+1)}}\vartheta_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$\cup \begin{pmatrix} \frac{\sqrt{m+1} - \sqrt{8m^2+m+1}}{2\sqrt{m(m+1)}}\vartheta_1 & \frac{\sqrt{m+1} - \sqrt{8m^2+m+1}}{2\sqrt{m(m+1)}}\vartheta_2 & \cdots & \frac{\sqrt{m+1} - \sqrt{8m^2+m+1}}{2\sqrt{m(m+1)}}\vartheta_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$\cup \begin{pmatrix} 0 \\ n(m-1) \end{pmatrix}$$

Hence,

$$\begin{aligned}
\mathscr{E}_{SC}(Spl_m(G)) &= \sum_{i=1}^{n} \left| \frac{\sqrt{m+1} + \sqrt{8m^2+m+1}}{2\sqrt{m(m+1)}}\vartheta_i + \frac{\sqrt{8m^2+m+1} - \sqrt{m+1}}{2\sqrt{m(m+1)}}\vartheta_i \right| \\
&= \sum_{i=1}^{n} |\alpha_i| \left(\frac{\sqrt{m+1} + \sqrt{8m^2+m+1}}{2\sqrt{m(m+1)}} + \frac{\sqrt{8m^2+m+1} - \sqrt{m+1}}{2\sqrt{m(m+1)}} \right) \\
&= \sum_{i=1}^{n} |\vartheta_i| \left(\frac{\sqrt{m+1} + \sqrt{8m^2+m+1} + \sqrt{8m^2+m+1} - \sqrt{m+1}}{2\sqrt{m(m+1)}} \right) \\
&= \sum_{i=1}^{n} |\vartheta_i| \left(\sqrt{\frac{8m^2+m+1}{m(m+1)}} \right) \\
&= \left(\sqrt{\frac{8m^2+m+1}{m(m+1)}} \right) \sum_{i=1}^{n} |\vartheta_i| \\
&= \left(\sqrt{\frac{8m^2+m+1}{m(m+1)}} \right) \mathscr{E}_{SC}(G)
\end{aligned}$$

Hence,

$$\mathscr{E}_{SC}(Spl_m(G)) = \left(\sqrt{\frac{8m^2+m+1}{m(m+1)}}\right)\mathscr{E}_{SC}(G)$$

□

Theorem 2.5. *For any graph G,*

$$\mathscr{E}_{SC}(G') = 2\mathscr{E}_{SC}(G)$$

Proof. Let G be a graph with n vertices. To obtained a double cover graph G' of the graph G by taking bipartition (X, Y), where $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ...y_n\}$ with two vertices x_i and y_j are adjacent in G' if and only if v_i and v_j are adjacent in G.
Then the sum-connectivity matrix of the graph G' is defined as

$$M_{SC}(G') = \begin{bmatrix} 0 & M_{SC}(G) \\ M_{SC}(G) & 0 \end{bmatrix}_{2n\times 2n}$$

where, $M_{SC}(G)$ is the sum-connectivity matrix of a graph G with eigenvalues are $\vartheta_1, \vartheta_2, ..., \vartheta_n$.
Then by Proposition 1.7,

$$\begin{aligned} spec_{SC}(G') &= spec(M_{SC}(G)) \cup spec(-M_{SC}(G)) \\ &= \begin{pmatrix} \vartheta_1 & \vartheta_2 & \cdots & \vartheta_n & -\vartheta_1 & -\vartheta_2 & \cdots & -\vartheta_n \\ 1 & 1 & & \cdots 1 & 1 & 1 & & \cdots 1 \end{pmatrix} \end{aligned}$$

Hence,

$$\begin{aligned} \mathscr{E}_{SC}(G') &= \sum_{i=1}^{n} |\vartheta_i| + \sum_{i=1}^{n} |-\vartheta_i| \\ &= 2\sum_{i=1}^{n} |\vartheta_i| \\ &= 2\mathscr{E}_{SC}(G) \end{aligned}$$

□

Corollary 2.6. *For any r-regular graph G,*

$$\mathscr{E}_{SC}(G') = \sqrt{\frac{2}{r}}\mathscr{E}(G)$$

Proof. Proof follows from the Theorem 2.1 and Theorem 2.5. □

3 Concluding Remarks

The energy of a graph is one of the important idea of spectral graph theory. This idea is a bond between chemical science and mathematical science. In this paper, we have obtained the relation between sum-connectivity energy of the graph and graph energy, also derived the sum-connectivity energy of m-shadow, m-splitting of a graph and double cover of a graph.

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